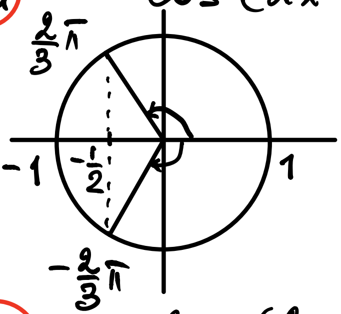
Lösningsskiss till tentan i 764G07, del 1, 2020-10-29

1a

$$\cos\left(2x+\frac{\pi}{2}\right) = -\frac{1}{2} \iff 2x+\frac{\pi}{2} = \pm\frac{2}{3}\pi + 2\pi n \iff$$

$$2x = -\frac{\pi}{2} \pm \frac{2}{3}\pi + 2\pi n \iff x = -\frac{\pi}{4} \pm \frac{\pi}{3} + \pi n$$



Svar: $x = \frac{\pi}{12} + \pi n, \quad x = -\frac{7}{12}\pi + \pi n.$

1b

$$\sin\left(2x+\frac{\pi}{4}\right) - \sin x = 0 \iff 2\sin\frac{x+\frac{\pi}{4}}{2}\cos\frac{3x+\frac{\pi}{4}}{2} = 0$$

$$\iff \sin\left(\frac{x}{2}+\frac{\pi}{8}\right) = 0 \text{ eller } \cos\left(\frac{3}{2}x+\frac{\pi}{8}\right) = 0 \iff$$

$$\frac{x}{2}+\frac{\pi}{8} = \pi n \text{ eller } \frac{3}{2}x + \frac{\pi}{8} = \frac{\pi}{2} + \pi n \iff 3x = \frac{3}{4}\pi + 2\pi n,\ x = \frac{\pi}{4} + \frac{2}{3}\pi n$$

Svar: $x = -\frac{\pi}{4} + 2\pi n, \quad x = \frac{\pi}{4} + \frac{2}{3}\pi n.$

1c

$$3\ln x - \ln(x+3) = \ln\frac{x^2}{x-1};$$

Krav: $\begin{cases} x>0 \\ x+3>0 \\ \frac{x^2}{x-1}>0 \end{cases} \Rightarrow x>1$

$$\Updownarrow$$

$$\ln\frac{x^3}{x+3} = \ln\frac{x^2}{x-1} \quad /\ln \text{ injektiv}/$$

$$\Rightarrow \frac{x^3}{x+3} = \frac{x^2}{x-1} \Rightarrow \frac{x^3}{x+3} - \frac{x^2}{x-1} = 0 \Rightarrow \frac{x^3(x-1)-x^2(x+3)}{(x-1)(x+3)} = 0$$

$$\Rightarrow x^2(x^2-x-x-3) = 0 \Rightarrow$$

$x = 0$ – falsk p.g.a. kravet

$x^2-2x-3=0 \iff (x+1)(x-3)=0$

$\Rightarrow$ $x=-1$ – falsk p.g.a. kravet

$x = 3$

Svar: $x = 3.$

2a

$$z^4+3z^3+2z^2-2z-4=0.$$

Sätt $p(z) = z^4+3z^3+2z^2-2z-4$. $p(1)=0 \Rightarrow$ (Faktorsats) $p(z) = (z-1)q(z)$

$$\begin{array}{l|l}
\multicolumn{1}{l}{z^3+4z^2+6z+4} & \\
\hline
z^4+3z^3+2z^2-2z-4 & z-1 \\
-(z^4-z^3) & \\
\hline
\quad 4z^3+2z^2-2z-4 & \\
\quad -(4z^3-4z^2) & \\
\hline
\qquad 6z^2-2z-4 & \\
\qquad -(6z^2-6z) & \\
\hline
\qquad\quad 4z-4 & \\
\qquad\quad -(4z-4) & \\
\hline
\qquad\qquad 0 &
\end{array}$$

$q(z) = z^3+4z^2+6z+4$

Vi ser att $q(-2)=0 \Rightarrow$

/enligt Faktorsatsen/

$q(z) = (z+2)h(z)$ där

$h(z) = z^2+2z+2 = (z+1)^2+1$

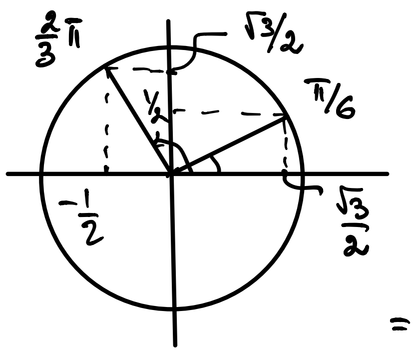
Alltså $p(z) = (z-1)(z+2)(z^2+2z+2) = 0 \iff$

$z_1 = 1,\ z_2 = -2\ ;\quad z^2+2z+2=0 \iff (z+1)^2+1=0 \iff$

$z_{3,4} = -1 \pm i$

Svar: $z_1 = 1,\ z_2 = -2,\ z_3 = -1+i,\ z_4 = -1-i.$

2b Räkna ut



$$z = \frac{(-1+i\sqrt{3})^5}{(\sqrt{3}+i)^4} = \Big/\ -1+i\sqrt{3} = 2\left(-\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right) = 2e^{i\frac{2}{3}\pi};\quad \sqrt{3}+i = 2\left(\tfrac{\sqrt{3}}{2}+i\tfrac{1}{2}\right) = 2e^{i\pi/6}\ \Big/ =$$

$$= \frac{(2e^{i\frac{2}{3}\pi})^5}{(2e^{i\pi/6})^4} = 2e^{i\left(\frac{10}{3}\pi - \frac{2}{3}\pi\right)} = 2e^{i\frac{8}{3}\pi} =$$

$$= 2e^{i(2\pi + \frac{2}{3}\pi)} = 2e^{i\frac{2}{3}\pi} =$$

$$= 2\left(\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\right) = 2\left(-\tfrac{1}{2}+i\tfrac{\sqrt{3}}{2}\right) = -1+i\sqrt{3}$$

Svar: $z = -1+i\sqrt{3}.$

3a
$$\lim_{x\to -1}\frac{x^3+2x^2+2x+1}{x^2+4x+3} = \left(\tfrac{0}{0}\right) \Big/ \text{Faktorsatsen} \Big/ \lim_{x\to -1}\frac{(x+1)(x^2+x+1)}{(x+1)(x+3)} =$$

$$= \lim_{x\to -1}\frac{x^2+x+1}{x+3} = \frac{1}{2}.$$

3b
$$\lim_{x\to\infty}\left(\sqrt{x^2+4x+1} - \sqrt{x^2-5x}\right) \overset{\text{konjugatreg.}}{=} \lim_{x\to\infty}\frac{(x^2+4x+1)-(x^2-5x)}{\sqrt{x^2+4x+1}+\sqrt{x^2-5x}} =$$

$$= \lim_{x\to\infty}\frac{9x+1}{\sqrt{x^2+4x+1}+\sqrt{x^2-5x}} = \lim_{x\to\infty}\frac{x\left(9+\frac{1}{x}\right)}{x\left(\sqrt{1+\frac{4}{x}+\frac{1}{x^2}}+\sqrt{1-\frac{5}{x}}\right)} = \frac{9}{2}.$$

3c
$$\lim_{x\to 1}\frac{\ln x}{\sin(2x-2)} \Big/ \begin{matrix} t = x-1 \\ t\to 0\end{matrix} \Big/ = \lim_{t\to 0}\frac{\ln(1+t)}{\sin 2t} =$$

$$= \lim_{t\to 0}\underbrace{\frac{\ln(1+t)}{t}}_{\to 1}\cdot\frac{t}{2t}\cdot\frac{1}{\underbrace{\frac{\sin 2t}{2t}}_{\to 1}} = \frac{1}{2}.$$

Svar: a) $\frac{1}{2}$, b) $\frac{9}{2}$, c) $\frac{1}{2}$.

4 $f(x) = e^{3x-x^3}$.

- $D_f = \mathbb{R}$
- $\lim\limits_{x\to-\infty} e^{3x-x^3} = \infty$, $\lim\limits_{x\to\infty} e^{3x-x^3} = 0 \Rightarrow$

$y=0$ – vågrät asymptot

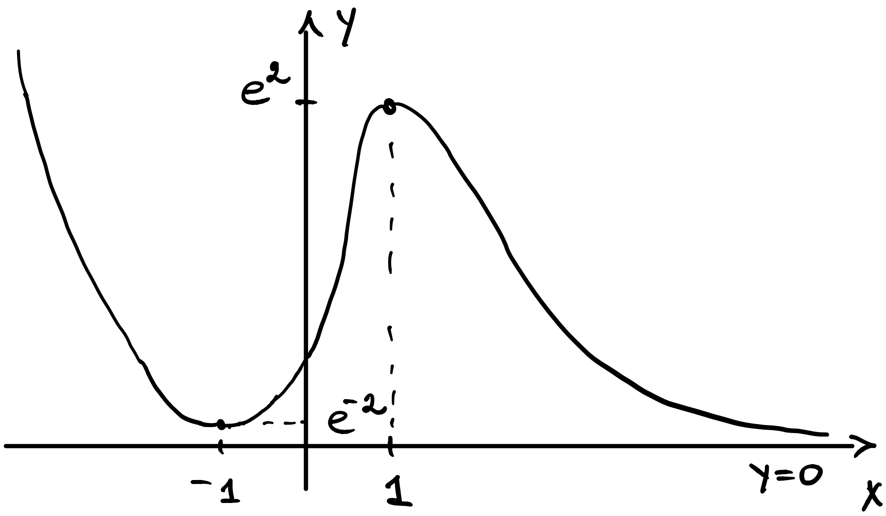
- $f'(x) = e^{3x-x^3}(3-3x^2) = 3e^{3x-x^3}(1-x^2) = 0 \Leftrightarrow$

$x = \pm 1$

$$\begin{array}{ccccc} - \searrow & & + \nearrow & & - \searrow \\ & -1 & & 1 & \\ & \text{lok. min} & & \text{lok. max} & \end{array}$$

$f(-1) = e^{-2}$

$f(1) = e^2$



Svar:

$y=0$ – vågrät asymptot

lok. min antas i p. $x=-1$,

$f(-1) = e^{-2}$

lok. max antas i p. $x=1$,

$f(1) = e^2$.

5 $f(x) = e^{2x} + x + 1$ är kontinuerlig och strängt växande p.g.a. $f'(x) = 2e^{2x}+1 > 0 \Rightarrow f^{-1}$ existerar och kontinuerlig.

f är deriverbar och $f'(x) = 2e^{2x}+1 \neq 0$ för alla $x \in \mathbb{R} \Rightarrow f^{-1}$ deriverbar i alla punkter $x \in \mathbb{R}$.

Vi ser att $f(0) = e^0 + 0 + 1 = 2 \Rightarrow$ enligt satsen

$$(f^{-1})'(2) = \frac{1}{f'(0)} = \frac{1}{3}$$

Svar: f deriverbar och $(f^{-1})'(2) = \frac{1}{3}$.

⑥ $\dfrac{xe^x}{(x+1)^2} = k$

Sätt $f(x) = \dfrac{xe^x}{(x+1)^2}$. För att svara på frågan ritar vi funktionskurva:

- $D_f =]-\infty, -1[\, \cup \,]-1, \infty[$
- $\lim\limits_{x\to-\infty} \dfrac{xe^x}{(x+1)^2} = 0, \quad \lim\limits_{x\to-1^-} \dfrac{xe^x}{(x+1)^2} = -\infty, \quad \lim\limits_{x\to-1^+} \dfrac{xe^x}{(x+1)^2} = -\infty,$

$$\lim_{x\to\infty} \frac{xe^x}{(x+1)^2} = \lim_{x\to\infty} \frac{xe^x}{x^2(1+\frac{1}{x})^2} = \lim_{x\to\infty} \frac{e^x}{x} \cdot \frac{1}{(1+\frac{1}{x})^2} =$$

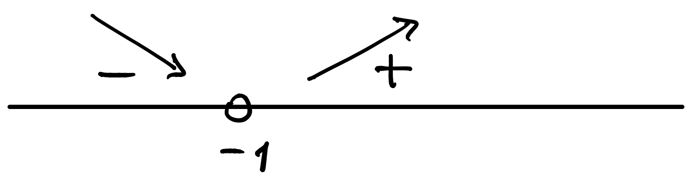
$$= \Big/ \begin{array}{l} \frac{e^x}{x} \to \infty \text{ p.g.a. hastighetstabell} \\ (1+\frac{1}{x})^2 \to 1 \text{ då } x \to \infty \end{array} \Big/ = \infty .$$

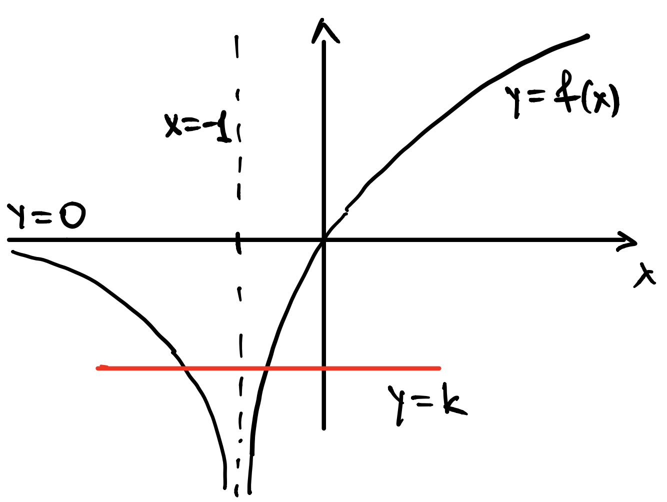
$\Rightarrow$ $y = 0$ – vågrät asymptot då $x \to -\infty$

$x = -1$ – lodrät asymptot.

- $f'(x) = \dfrac{(e^x + xe^x)(x+1)^2 - 2(x+1)xe^x}{(x+1)^4} = \dfrac{e^x((1+x)^2 - 2x)}{(x+1)^3} =$

$$= \frac{e^x(1+x^2)}{(x+1)^3} \neq 0$$





Svar: Ekvationen har
2 lösn. för $-\infty < k < 0$
1 lösn. för $k \geq 0$.

7

$$f(x)=\begin{cases} 5x^2+x^4\cos\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

Vi beräknar först $f'(x)$:

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{5h^2+h^4\cos\frac{1}{h}}{h}=\lim_{h\to 0}\left(5h+h^3\cos\frac{1}{h}\right)=$$

$= \Big/ \underbrace{h^3}_{\to 0 \text{ då } h\to 0}\ \underbrace{\cos\frac{1}{h}}_{\text{begr.}} \to 0 \Big/ = 0.$ Samt för $x\neq 0$ har vi $\left(5x^2+x^4\cos\frac{1}{x}\right)'=$

$= 10x+4x^3\cos\frac{1}{x}-x^4\sin\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right).$ Alltså

$$f'(x)=\begin{cases} 10x+4x^3\cos\frac{1}{x}+x^2\sin\frac{1}{x}, & x\neq 0 \\ 0, & x=0 \end{cases}$$

$\Rightarrow$ enligt definition

$$f''(0)=\lim_{h\to 0}\frac{f'(0+h)-f'(0)}{h}=\lim_{h\to 0}\frac{10h+4h^3\cos\frac{1}{h}+h^2\sin\frac{1}{h}-0}{h}=$$

$= \lim_{h\to 0}\left(10+4h^2\cos\frac{1}{h}+h\sin\frac{1}{h}\right) = \Big/ \underbrace{h^2}_{\to 0}\ \underbrace{\cos\frac{1}{h}}_{\text{begr.}} \to 0,\ \ \underbrace{h}_{\to 0}\ \underbrace{\sin\frac{1}{h}}_{\text{begr.}} \to 0 \Big/ = 10.$

Svar: $f''(0)=10.$